On the complexity of Commuting Local Hamiltonians, and tight conditions for Topological Order in such systems

Dorit Aharonov
School of Computer Science and Engineering
The Hebrew University
Jerusalem, Israel

Lior Eldar
School of Computer Science and Engineering
The Hebrew University
Jerusalem, Israel

Abstract— The local Hamiltonian problem plays the equivalent role of SAT in quantum complexity theory. Understanding the complexity of the intermediate case in which the constraints are quantum but all local terms in the Hamiltonian commute, is of importance for conceptual, physical and computational complexity reasons. Bravyi and Vyalyi showed in 2003 [10], using a clever application of the representation theory of C*-algebras, that if the terms in the Hamiltonian are all two-local, the problem is in NP, and the entanglement in the ground states is local. The general case remained open since then. In this paper we extend this result beyond the two-local case, to the case of three-qubit interactions. We then extend our results even further, and show that NP verification is possible for three-wise interaction between qutrits as well, as long as the interaction graph is planar and also "nearly Euclidean" in some well-defined sense. The proofs imply that in all such systems, the entanglement in the ground states is local.

These extensions imply an intriguing sharp transition phenomenon in commuting Hamiltonian systems: the ground spaces of 3-local "physical" systems based on qubits and qutrits are diagonalizable by a basis whose entanglement is highly local, while even slightly more involved interactions (the particle dimensionality or the locality of the interaction is larger) already exhibit an important long-range entanglement property called Topological Order. Our results thus imply that Kitaev's celebrated Toric code construction is, in a well defined sense, optimal as a construction of Topological Order based on commuting Hamiltonians.

1. Introduction

The problem of approximating the ground energy of a local Hamiltonian describing a physical system is one of the major problems in condensed matter physics; in the area of quantum computation this problem is called the local Hamiltonian problem [15]. Formally, in the k-local Hamiltonian problem, we are given a Hamiltonian H which is a sum of positive semidefinite terms, each acting on a set of at most k out of n qubits, where k is of order 1, and each term is of bounded norm. Moreover, we are given two numbers, $b > a$ such that $b - a \geq \frac{1}{poly(n)}$. We are asked whether H has an eigenvalue below a or all its eigenvalues are above b, and we are promised that the instance belongs to one of the two cases.

It turns out that the problem of understanding ground states and ground values of local Hamiltonians, central to condensed matter physics, is the quantum generalization of one of the most important problems in classical computational complexity, namely, SAT. Indeed, in a seminal work, Kitaev has shown that in parallel to the importance of the SAT problem in NP theory, the local Hamiltonian problem is complete for the quantum analogue of NP (denoted QMA) in which both witness and verifier are quantum rather than classical. The analogy between the quantum and the classical problems is derived by viewing the terms of the Hamiltonians as generalizing the notion of classical constraints; energies are viewed as a penalty for a constraint violation. For example, to view the local constraints for the classical SAT as a special instance of local Hamiltonians, we assign for each clause a projection on the assignment forbidden by this clause. The projections we derive are all diagonalizable in the computational basis; in the general local Hamiltonian problem, the terms need not be diagonal in any particular basis, and the ground state can be highly entangled. This connection linking the physics and the computational complexity problems has drawn much attention over the past few years, and has led to many exciting results and insights (eg., [15], [13], [1], [16], [5], [10]).

The computational view of the local Hamiltonian problem and its connection to classical NP problems, has led Bravyi and Vyalyi in [10] to the following very natural question: what would happen if we only generalize from classical to quantum "half way": we allow the terms in the Hamiltonian to be projections in any basis, but we restrict them in that all the projections pairwise commute. We are asked to decide whether the ground energy is 0 (namely, there exists a state which is in the ground space of all projections) or it is larger than 0 (for pairwise-commuting projections, the overall energy, namely eigenvalue, of such a state must be at least 1). This problem is the commuting local

Hamiltonian problem [1].

The interest in the commuting Hamiltonian problem is related to several important issues in quantum computational complexity. The first is conceptual: a common intuition is that the counter intuitive phenomena in quantum mechanics stems from the fact that non-commuting operators are involved (cf the Heisenberg's uncertainty principle). One might conjecture, using this intuition, that the commuting local Hamiltonian problem is far weaker than the general local Hamiltonian problem, and might be of the same complexity as SAT, namely, lie in NP. However, a counter intuition exists: The intriguing strictly quantum phenomenon of Topological Order, which is exhibited for example in Toric codes [14], can be achieved by ground states of commuting Hamiltonians. It is thus natural to ask where does the computational complexity of the commuting Hamiltonian problem lie: is it in NP, is it perhaps quantum-NP complete (where here the relevant quantum analogue of NP is in fact, QMA_1, where there is only one sided error) or maybe the commuting local Hamiltonian problem defines an intermediate computational class of its own?

The study of this problem can also be viewed as tightly related to an exciting major open problem in quantum Hamiltonian complexity: the question of whether a PCP-like theorem holds in the quantum setting or not [2]. Embarrassingly, this problem is still open even for the seemingly much easier case of commuting local Hamiltonians. Clearly, a PCP-type theorem would follow trivially if the commuting local Hamiltonian problem were in NP, but even if this were not true, one might still hope to prove a PCP-type theorem for the restricted problem before proceeding to the more general case. We recall that several results in quantum Hamiltonian complexity, such as the area law in 1Dim [11] the decay of correlations in gapped Hamiltonians [12], and quantum gap amplification [2] were all proven by starting from the easier commuting case, and generalizing from there; it seems reasonable to hope that better understanding of the commuting case would help clarify the quantum PCP conjecture in general. More generally, it seems that understanding the complexity of the commuting local Hamiltonian problem will necessarily require new insights regarding the nature of multi-particle entanglement.

In [10] an important step was made towards resolving the computational complexity of the commuting local Hamiltonian problem. Bravyi and Vyalyi showed that for two-body interactions, regardless of the dimensionality d of the particles involved, the problem lies in NP. To do this they cleverly apply the theory of representations of C^*-algebras to the problem. However, their methods break down for three-wise interactions. The general problem was thus left open by [10], and no progress was noted on this problem since then.

[1] We note that this problem is equivalent to the more general case when the terms can be taken as positive-semidefinite commuting operators, since for such an input one can replace each local term with projections on the non-zero eigenspaces of that term.

Before we state our results, let us recall briefly the methods of Bravyi and Vyalyi and explain why they fail in the case of three particle interactions. Consider the hypergraph describing the interactions in the Hamiltonian. We observe that in the two-local case, every particle is the center of a "star" of interactions - the interactions acting on q intersect only on q. Bravyi and Vyalyi prove a lemma (restated and reproved in a simpler way here, Lemma 3.4), which shows that particles which are centers of "stars", are what we call "separable". This means that if q is such a center of a star, its Hilbert space $\mathcal{H}_q$ can be *decomposed* to a direct sum of subspaces, which are all preserved by all interactions involving q:

$$\mathcal{H}_q = \bigoplus_{\alpha} \mathcal{H}^q_\alpha.$$

Moreover, each subspace $\mathcal{H}^q_\alpha$ can be written as a tensor product of sub-particles, such that when restricting attention to one of the invariant subspaces $\mathcal{H}^q_\alpha$, each particle q interacts with a different sub-particle of q! When all particles are center of stars as in the two-local case, after each particle is restricted to one of its subspaces the restricted Hamiltonian is a set of disjoint edges.

From this [10] derive a proof that the two-local problem lies in NP - essentially, the witness is the specification of the choice α of the correct subspace of each particle, in which the groundstate lies. Their proof also implies that in the two-local case, there is an eigenbasis of the Hamiltonian in which any eigenstate (and in particular any ground state) has a very limited and local structure of entanglement - the state can be generated by a depth-two quantum circuit which uses only two-local gates. Of course, a natural question is whether these techniques can be applied for the more general case, namely, for higher values of k.

Trivially, when generalizing from 2-local interactions to 3-local interactions we immediately loose the star topology See, for example, Figure 1.

However, this example is not truly a problem when we restrict our attention to qubits, since the low dimensionality implies that one cannot "block-diagonalize" an operator on a qubit q in more than one way. Thus it turns out that in the example above, there is indeed a "consensus" decomposition of q preserved by all 3 operators on q. However, consider the example of 4 operators on 4 qubits in Figure 2.

Since any pair of operators share 2 qubits, it may be the case that no single qubit has a direct-sum decomposition which is preserved by all operators on

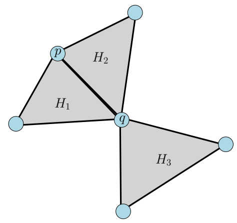

Figure 1. In the example both H_1 and H_2 share a single qubit q with H_3. By the methods of Bravyi and Vyalyi, H_1, and H_3 agree on some decomposition of q, and so do H_2 and H_3. Yet, because H_1 and H_2 share two qubits p and q, they do not agree necessarily on the same decomposition of q.

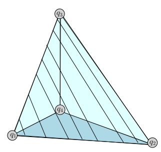

Figure 2. An example of a topology of interactions which can be defined in such a way that, say, for q_1, no decomposition exists, which is preserved by all operators acting on it.

that qubit. This in fact emanates from the nature of the commutativity relation for 3-local terms: commutativity can emerge from considering not just one particle as in [10] but may involve more complex relations involving two particles.

1.1. Results: The Complexity of 3-local Commuting Hamiltonians of qubits and qutrits

In this paper we extend the results of [10] to three-local interactions with the following two results.

Theorem 1.1: The problem of 3-local commuting Hamiltonian on qubits is in NP.

Theorem 1.2: The problem of 3-local commuting Hamiltonian on qutrits is in NP, as long as the interaction graph is planar and *nearly Euclidean*.

In the latter Theorem, the interaction graph is the graph whose nodes are the particles, and an edge exists iff its two nodes participate in one interaction term in the Hamiltonian. The notion of "Nearly Euclidean" formalizes the requirement that the embedding in the plane makes sense physically: no area on the plane can have a particularly high density of particles, and only close-by particles can interact. This of course includes also the interesting special case of periodic lattices, or small perturbations of those.

Unlike what might have been expected, the proofs of the above theorems do not seem to follow easily from the result of [10], and are in fact quite involved. The way we overcome the obstacles mentioned above is by showing that complex structures, such as the example of Figure 2, cannot be overly complicated; once we remove all separable qudits from the system (namely, trivial qudits for which the methods of Bravyi and Vyalyi [10] apply) the interaction graph of the residual system is subject to severe geometrical constraints. These constraints enable coarse graining the remaining particles so that the induced interactions are guaranteed to be two-local, and the methods of [10] can be applied. The technical details are very different for the two theorems; In this extended abstract we will only be able to provide a general outline of the main steps in the two proofs. The web version [3] contains the full proofs.

1.2. Results: Tight conditions on Topological Order

Topological Order is a purely quantum phenomenon related to long range entanglement, which has captured much attention in the context of quantum fault-tolerance and possible implementations; Roughly, a state exhibits a Topological Order if there exists a state orthogonal to it, and the two cannot be distinguished or connected by a local operator. A celebrated example is Kitaev's Toric Code [14]; it can be defined as the ground space of a set of 4-local commuting operators on qubits arranged on a two dimensional grid. Topological Order defined via commuting local Hamiltonians is particularly interesting; recently it has been shown [6][7] that such systems are resilient to local perturbations. It is therefore natural to ask whether it is possible to achieve Topological Order in ground states of local commuting Hamiltonians, with smaller dimensionality or with less particles interacting than in the Toric code construction. Using the above results, we resolve this problem to the negative. We show that Kitaev's construction is optimal in a well defined sense.

To understand how our results are related to conditions on Topological Order, observe the following. A key property of Topological Order states is that their entanglement is non-local. In particular, Bravyi, Hastings and Verstraete showed in [8], that if a nearest neighbor quantum circuit generates a state with Topological Order on the $n \times n$ grid, the circuit has to be of depth $\Omega(\sqrt{n})$. The methods we use, as well as those of [10], however, imply that the ground space of systems for which Theorems 1.1 and 1.2 apply has an orthonormal basis of states with localized entanglement; more precisely, an orthonormal basis of states each of which can be generated by a constant depth nearest-neighbor circuit. This means that such systems cannot exhibit Topological Order in all the states in their groundspace. We know however, that 3-local Hamiltonians on qudits of dimension 4 *can* exhibit Topological Order, since the Toric code can be seen as such a system (by gluing pairs of nearest-neighbor qubits together.) We prove:

Theorem 1.3: **Tight conditions for Topological Order** (Roughly) Consider a system of particles with commuting interactions which are either 2-local, or they are 3-local and the dimensionality of the particles is at most 3. Moreover, assume the interaction graph is Nearly

Euclidean planar. Then this system cannot exhibit Topological Order, and moreover, in a well defined sense, the entanglement in the ground space is local. On the other hand, there exist nearly Euclidean planar systems of 3-local interactions with particles of dimensionality 4 that exhibit Topological Order.

We thus derive a tight boundary between local entanglement and Topological Order. We deduce that Kitaev's construction cannot be simplified either in terms of particle dimensionality or number of particles in each interaction, and so it is optimal for commuting Hamiltonians constructions of Topological Order.

1.3. Conclusions and Further work

The results in this paper focus on two aspects of commuting Hamiltonians: the first is extending the containment in NP also for three body interactions, where a fundamental barrier is encountered exactly when Topological Order can be present in the ground space. Three body interactions seemed before as the barrier standing between [10] and the extension towards a proof of containment in NP of the general case; here we show that the barrier is far more intriguing, and has to do with the appearance of Topological Order.

The second aspect is the proof that Kitaev's celebrated construction of Topological Order using Toric codes is optimal, a statement which is of interest in various contexts, such as physical implementations of Topological Order states, topological quantum codes, and the understanding of multiparticle entanglement.

The barrier exposed in this paper is by no means an indication that the general commuting local Hamiltonian problem is not in NP. In fact, we hope that the barrier encountered here would clarify how we should proceed in order to resolve the question.

An interesting first step in this direction was made recently by Schuch [17], following the first publication of the results presented here. Schuch showed that the commuting local Hamiltonian problem with *four*-local nearest-neighbor interactions between qubits on a planar grid is in NP. The proposed NP protocol in ([17]) does not involve the verifier holding a description of a short circuit generating the eigenstate of the system, as this would contradict the lower-bound on the circuit depth of the Toric Code (1.3). In fact, the verifier in Schuch's result does not hold any kind of description of the groundstate of the system, and is convinced that there exists a mutual groundstate without being able to actually "hold" one. It is wide open whether such implicit verifications can be extended to more general commuting Hamiltonian systems.

Another possible direction to explore is the following. As is well known, Topological Order states such as Toric codes do have short classical descriptions, which are in fact classical descriptions of small depth quantum circuits, except those circuits are *non-local* (i.e., not nearest neighbor on the grid). These are called MERA [19], [4]. Those descriptions allow computing local observables efficiently using a classical computer. From the point of view of NP verification, this is clearly sufficient. If one can show that such MERA descriptions exist for any ground state of commuting Hamiltonian, this would imply that the problem lies in NP. It is possible that the methods of [10] can be used in an innovative way (perhaps by recursion or by other means) to imply that there exist such MERA-type poly-size classical descriptions of eigenstates for any k-local commuting Hamiltonian.

Though the bound on the size of the entanglement structures that are exhibited in our proofs is constant, this constant is much larger than the natural scales of the system (say, 2 or 3 particles). Is this a true property of the systems we consider, or just an artifact of our proof methods?

Finally, a technicality in the proof of the qutrit case is that the graph is required to be nearly Euclidean, rather than just planar. We speculate that this requirement can be removed; This does not have strong implications for the results, since Topological Order is in any case studied in nearly Euclidean systems, but it would be nice to close that corner and make our statements cleaner.

Organization of rest of extended abstract We start in Section 2 by notations and mathematical background; We proceed in Section 3 to provide a simplified proof of the result of Bravyi and Vyalyi, which is the basis for the rest of the paper, in the last two sections we provide sketches for our two main contributions, namely, for the proof of theorem (1.1) in Section (4) and for the proof of theorem (1.2) in (5).

2. Background, Notations & Definitions

2.1. Hamiltonians and Hilbert Spaces

We use the following standard notation. We denote Hilbert spaces by graphical symbols: $\mathcal{H}, \mathcal{H}_i$, etc. The set of linear operators over the complex numbers, acting on a given Hilbert space $\mathcal{H}$, is denoted by $L(\mathcal{H})$. Denote by $\mathcal{H}$ the Hilbert space of n qudits: $\mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_n$. The dimensionality of the qudits is denoted by d. A k-local operator h is an operator which acts on a subset of size k of the n qudits $S \subseteq \{1, \ldots, n\}$, hence $|S| = k$, and we have $H \in \mathbf{L}(\otimes_{j\in S}\mathcal{H}_j) \otimes \left(\otimes_{j\notin S} I_j\right)$.

2.2. Hamiltonians acting on qubits

To specify that an operator H acts non-trivially on some specific qubit q, we write $H(q)$. We say an operator acts non-trivially on a particle q if the operator cannot be written as a tensor product of the identity operator on q and some operator on the remaining particles. The set of qudits examined non-trivially by an operator H_i is denoted by A_i; The set of particles examined non-trivially by a set of operators B, is denoted by A_B.

2.3. Local Hamiltonian Problem and interaction graph

Definition 2.1: **The (k, d) local Hamiltonian problem for commuting operators, on n qudits of dimension d, denoted** $CLH(k, d)$ is defined as follows. We are given a set S of $poly(n)$ k-local projections, H_i, acting on n particles each of dimension d, such that all terms in S pairwise commute. We are asked whether there exists an eigenstate of $H = \sum_{i \in S} H_i$ with eigenvalue 0 or not.

Definition 2.2: **The Interaction graph of an instance S of $CLH(k, d)$** is the graph $G_S = (V, E)$, where V is a set of n nodes, each corresponding to a qudit, and an edge connecting nodes i and j is in E (namely, $(i, j) \in E$) if there exists some $H_m \in S$ such that both $i, j \in A_m$.

2.4. Operators Preserving Subspaces

An operator A is said to preserve a subspace S if $A(S) \subseteq S$. The following facts are trivial to prove:

Fact 2.3: If A is Hermitian, then it preserves a subspace S iff it preserves the orthogonal complement of S.

Fact 2.4: If a linear operator A commutes with a projection on a subspace S, then A preserves S.

2.5. Algebras

In this paper we consider finite dimensional C^*-algebras, denoted by $\mathcal{A}$, $\mathcal{B}$ etc. For the purposes of this paper, it suffices to consider the case of $\mathcal{A} \subseteq L(\mathcal{H})$, i.e., algebras of linear operators (described by matrices) with the additional restriction that $\mathcal{A}$ is closed under the adjoints (i.e, the *dagger* operation).

The algebra generated by a set of matrices (always of the same dimensionality) is defined either as the minimal algebra that contains the linear subspaces spanned by the generators, or equivalently, the algebra generated by the set of generators union with the identity matrix.

2.6. Algebras induced by operators

Definition 2.5: **Algebra induced by an operator:** Let $H = H(q)$ be an operator on q, and let us write

$$H = \sum_{\alpha} A_\alpha \otimes B_\alpha \tag{1}$$

such that A_α acts on q, and B_α acts on the rest of the environment, and the set $\{B_\alpha\}$ is linearly independent. Then the *algebra induced by H on q* is the algebra inside $\mathcal{L}(\mathcal{H}_q)$ generated by $\{A_\alpha\}_\alpha \cup \{I\}$.

Fact 2.6: Given an operator $H(q)$, the induced algebra on q, $\mathcal{A}_q^H$ is independent of our choice of how to write H as a sum as in Equation 1, so long as the B_α operators are linearly independent.

Proof: The elementary proof is omitted here. ■

Fact 2.7: Given a Hermitian operator $H(q)$, the induced algebra on q is closed under the adjoint operator.

Proof: We write $H = \sum_\alpha A_\alpha \otimes B_\alpha$ with B_α linearly independent; then the induced algebra is the one generated by $\{A_\alpha\}_\alpha \cup \{I\}$. But since H is Hermitian, $H = \sum_\alpha A_\alpha^\dagger \otimes B_\alpha^\dagger$ and so the induced algebra is also the algebra generated by $\left\{A_\alpha^\dagger\right\}_\alpha \cup \{I\}$ by Fact 2.6. This means that the induced algebra also contains the adjoint of the generators, and hence is closed under the adjoint. ■

A simple but crucial fact to this paper, whose elementary proof is omitted, is the following:

Fact 2.8: Consider two commuting Hamiltonian terms $H_{j,k}$ intersecting only on the qudit j. Then the algebras $\mathcal{A}_{j.k}$ induced by these operators on j commute with each other.

2.7. Representation theory of algebras

Definition 2.9: **Center of a C^*-algebra, $Z(\mathcal{A})$:** The center of an algebra $\mathcal{A}$ is defined to be the set of all operators in $\mathcal{A}$ which commute with all the elements in $\mathcal{A}$. It is denoted by $Z(\mathcal{A})$.

Definition 2.10: **A reducible / irreducible C^*-algebra:** An algebra $\mathcal{A}$ is said to be irreducible if its center is trivial, i.e. $\mathbf{Z}(\mathcal{A}) = c \cdot I$, and otherwise it is reducible.

A well-known decomposition theorem from the representation theory of C^*-algebras states [20]:

Fact 2.11: Let $\mathcal{A}$ be a C^*-algebra on some Hilbert space $\mathcal{H}$. Then, there exists a decomposition of $\mathcal{H}$ into a direct sum of orthogonal subspaces $\mathcal{H}_\alpha$, where each $\mathcal{H}_\alpha$ is a tensor product of two Hilbert spaces, $\mathcal{H}_\alpha = \mathcal{H}_\alpha^1 \otimes \mathcal{H}_\alpha^2$ such that

$$\mathcal{A} \approx \bigoplus_\alpha \mathbf{L}\left(\mathcal{H}_\alpha^1\right) \otimes \mathbf{I}\left(\mathcal{H}_\alpha^2\right).$$

The projections on the subspaces $\mathcal{H}_\alpha$ generate $\mathbf{Z}(\mathcal{A})$, and for each subspace $\mathcal{H}_\alpha$ the algebra $\mathcal{A}_\alpha$ (which is defined to be the algebra $\mathcal{A}$ restricted to $\mathcal{H}_\alpha$), is irreducible.

Claim 2.12: Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two commuting algebras on a Hilbert space $\mathcal{H}$, and let $\mathcal{A}_1$ be decomposed as in fact (2.11) - i.e. a decomposition $\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$ such that $\mathcal{A}_1 \approx \bigoplus_\alpha \mathbf{L}\left(\mathcal{H}_\alpha^1\right) \otimes \mathbf{I}\left(\mathcal{H}_\alpha^2\right)$. Then $\mathcal{A}_2$ preserves each subspace $\mathcal{H}_\alpha$ in the decomposition above.

Proof: By Fact 2.11 for each subspace $\mathcal{H}_\alpha$ there exists a projection $\Pi_\alpha \in \mathbf{Z}(\mathcal{A}_1)$ whose image is $\mathcal{H}_\alpha$. Since $\mathcal{A}_1$ and $\mathcal{A}_2$ commute, then each projection Π_α commutes with all operators in $\mathcal{A}_2$. Thus, by Fact 2.4 $\mathcal{A}_2$ preserves $\mathcal{H}_\alpha$ for all α. ■

3. 2-LOCAL CLH IS IN NP (REVISED FROM [10])

Theorem 3.1: $CLH(2, d)$ is in NP for any constant dimension d.

We start by using the basic facts from the theory of representations of algebras presented in Section 2 to prove an important lemma.

Definition 3.2: **Separating Decomposition** Let $\{\mathcal{A}_j\}_{j=1}^k$ be k mutually commuting algebras on some Hilbert space $\mathcal{H}$. A separating decomposition is a direct-sum decomposition of $\mathcal{H}$:

$$\mathcal{H} = \bigoplus_\alpha \mathcal{H}_\alpha$$

that is preserved by all algebras, such that

$$\mathcal{H}_\alpha = \mathcal{H}_\alpha^0 \otimes \mathcal{H}_\alpha^1 \otimes \mathcal{H}_\alpha^2 \otimes \cdots \otimes \mathcal{H}_\alpha^k$$

and for all $1 \leq j \leq k$ we have:

$$\mathcal{A}_j|_{\mathcal{H}_\alpha} \approx I_{\mathcal{H}_\alpha^0} \otimes I_{\mathcal{H}_\alpha^1} \cdots \otimes I_{\mathcal{H}_\alpha^{j-1}} \otimes \mathbf{L}(\mathcal{H}_\alpha^j) \otimes I_{\mathcal{H}_\alpha^{j+1}} \cdots \otimes I_{\mathcal{H}_\alpha^k}.$$

Claim 3.3: Let $\{\mathcal{A}_j\}_{j=1}^k$ be k mutually commuting algebras on some Hilbert space $\mathcal{H}$. There exists a separating decomposition of $\mathcal{H}$.

Proof: Suppose that the algebra $\mathcal{A}_j$ are all irreducible algebras - i.e. have trivial centers. Using fact (2.11) we have that each $\mathcal{A}_j$ is isomorphic to the full set of linear operators on some subsystem of $\mathcal{H}$. In other words, $\mathcal{H} = \mathcal{H}^j \otimes \mathcal{H}^{rest}$ and

$$\mathcal{A}_j \approx \bigoplus_\alpha \mathbf{L}\left(\mathcal{H}_\alpha^j\right) \otimes \mathbf{I}\left(\mathcal{H}_\alpha^{rest}\right).$$

Consider first $\mathcal{A}_1$ and $\mathcal{H}_\alpha^1$. Since the algebras commute, each $\mathcal{A}_j$ for $j > 1$ must act as the identity on the $\mathcal{H}_\alpha^1$, and hence acts not trivially only on $\mathcal{H}_\alpha^{rest}$; We proceed by induction, to derive that each $\mathcal{A}_j$ is isomorphic to the full set of linear operators on a separate sub-particle and thus the lemma follows in this case.

Now we generalize to the case where at least one algebra is reducible. Let us examine the algebra $\mathcal{A}$ generated by the set of all operators on the particle $\mathcal{H}$, that commute with any $A \in \mathcal{A}_j$ for all j. It is easy to check that since the $\mathcal{A}_j$ are closed under adjoint (by Fact 2.7) then so is $\mathcal{A}$. By fact (2.11) algebra $\mathcal{A}$ admits a decomposition, such that inside each subspace, it is isomorphic to the full set of operators on some subsystem, tensor with identity. By Claim (2.12) this decomposition is preserved by all $\mathcal{A}_j$ since they commute with $\mathcal{A}$.

We consider the algebras $\mathcal{A}_j$ restricted to these subspaces. We want to show that these restricted algebras are all irreducible. This follows since it turns out that the center of the algebra $\mathcal{A}$ in fact contains the centers of the algebras $\mathcal{A}_j$. To show this, first notice that $\mathbf{Z}(\mathcal{A}_j) \subseteq \mathcal{A}$. This is true, because any element of $\mathbf{Z}(\mathcal{A}_j)$ commutes with any element of $\mathcal{A}_j$ by definition algebra center, and it also commutes with any element from any other algebra $\mathcal{A}_{j'}$ for $j \neq j'$ since the algebras $\mathcal{A}_j$ and $\mathcal{A}_{j'}$ commute for any j'. In fact, since $\mathbf{Z}(\mathcal{A}_j)$ commutes with all the generators of $\mathcal{A}$, it is also contained in the center of $\mathcal{A}$. So $\mathbf{Z}(\mathcal{A}_j) \subseteq \mathbf{Z}(\mathcal{A})$.

Therefore, since $\mathcal{A}$ is irreducible inside each of the subspaces, so are $\mathcal{A}_j$. So the decomposition of the algebra $\mathcal{A}$ decomposes each algebra $\mathcal{A}_j$, into irreducible commuting algebras, which by the first paragraph must act on separate subsystems inside each subspace. ∎

We are now ready to prove the following crucial fact:

Lemma 3.4: **Decomposition of the center of a star (The star lemma) [adapted from [10]].** Let S be an instance of $CLH(2,d)$ whose interaction graph is a star: this means that there is a particle j, and each 2-local $H_{j,k}$ examines j and another particle k (where different terms act on different k's). Then there exists a direct sum decomposition

$$\mathcal{H}^j = \bigoplus_\alpha \mathcal{H}_\alpha^j \tag{2}$$

such that inside each subspace $\mathcal{H}_\alpha^j$ there appears a tensor product structure

$$\mathcal{H}_\alpha^j = \mathcal{H}_\alpha^{j.j} \otimes \left(\bigotimes_{(j,k)\in E} \mathcal{H}_\alpha^{j.k} \right) \tag{3}$$

where k runs over all other particles, such that all operators $H_{j,k}$ preserve the subspaces $\mathcal{H}_\alpha^j$, and moreover,

$$H_{j,k}|_{\mathcal{H}_\alpha^j} \in \bigotimes_{l\neq k} I_{\mathcal{H}_\alpha^{j.l}} \otimes \mathbf{L}\left(\mathcal{H}_\alpha^{j.k} \otimes \mathcal{H}^k\right) \tag{4}$$

Proof: We write each Hamiltonian as a sum of tensor product terms. $H_{j,k} = \sum_\alpha A_\alpha^k \otimes B_\alpha^k$, where A_α^k acts on $\mathcal{H}_j$ and B_α^k acts on $\mathcal{H}_k$, and the operators $\left\{B_\alpha^k\right\}_\alpha$ are linearly independent. We consider the C^*-algebra generated by $\{A_\alpha^k\}_\alpha \cup \{I\}$, and denote it $\mathcal{A}_{j.k}$. The key point is that any pair of $\mathcal{A}_{j.k}$ algebras commute, due to Fact (2.8). We can therefore apply claim (3.3) and this implies the result. ∎

From the lemma above, we now prove the result of [10]. The main insight is that in the two local case, for any qudit, the interactions involving that qudit form a star, so lemma 3.4 can be applied. Merlin helps Arthur find the groundstate by providing him with the correct index α in the decomposition of each particle.

Proof: (of Theorem 3.1) For an instance S of $CLH(2,d)$ the interaction graph is a star w.r.t. each qudit of the system. Hence Lemma (3.4) applies to each qudit separately. This implies that for each qudit there exists a decomposition to a direct sum of subspaces, so that the different terms in the Hamiltonian involving that qudit, act on separate sub-particles within each subspace. Therefore, there exists a decomposition of the entire Hilbert space into orthogonal subspaces, that are preserved by all local terms of S, such that first, each subspace is merely a tensor product of per-qudit subspaces. More importantly, inside each such subspace, the two-local terms do not intersect. S has eigenvalue 0 if and only if there exists a subspace M_0 (of the space of all qudits), such that all local terms in the Hamiltonian, when restricted to M_0, have a zero eigenspace. Merlin and Arthur find this decomposition separately, and then Merlin sends Arthur a description of M_0, indexed by some canonical convention, and Arthur verifies that the each local term, when restricted to M_0, has a non degenerate zero eigenspace. ■

4. Three-wise interactions of qubits

We provide here the proof of Theorem 1.1, omitting some of the details for lack of space. The first step in the proof is to use the tools of [10] to identify and remove seperable qubits.

Definition 4.1: **Separable qubit.** Given an instance S of $CLH(3,2)$, a qubit q is said to be separable if there exists a direct-sum decomposition of its Hilbert space to two one dimensional spaces,

$$\mathcal{H}_q = \bigoplus_{\alpha\in\{0,1\}} \mathcal{H}_q^\alpha$$

such that any operator $H(q)$ in S which acts on q preserves this decomposition:

$$H(q) = \bigoplus_{\alpha} H(q)|_{\mathcal{H}_q^\alpha},$$

where $H(q)|_{\mathcal{H}_q^\alpha}$ is the restricted projector. Observe that the restricted projection in this case is also a projection.

In the case of qubits, when a non-trivial decomposition exists, it must be into two subspaces of dimension one each; when restricting to one such subspace, the state of the qubit becomes some tensor product state with the rest of the system. This means that those qubits can in fact be removed from the system since Merlin can provide their state separately. We have thus reduced the problem to a problem in which all qubits are non-separable.

We now present the most important component of the proof, which is the characterization of the geometric properties of the interaction graph, after the removal of separable qubits. We treat each connected component separately, so we may assume the graph is connected.

4.1. Geomteric Properties

Given an instance H of $CLH(3,3)$, let $G_H = (V, E)$ be its interaction graph.

Definition 4.2: **Butterfly.** A $\triangleright\triangleleft$ (butterfly) relation with respect to q between two operators acting on the same qubit q, $H_1(q)$, $H_2(q)$ is said to holds if $A_1 \cap A_2 = \{q\}$. We denote this by $H_1 \triangleright\triangleleft H_2$.

A $\triangleright\triangleleft$ relations yields a direct-sum decomposition which is preserved by both operators. Formally stated:

Claim 4.3: For any pair of operators acting non trivially on q, H_1 and H_2, with $H_1 \triangleright\triangleleft H_2$ with respect to q, there exists a non-trivial decomposition of $\mathcal{H}_q$ into a sum of two one dimensional subspaces which are preserved by both operators.

Proof: Denote by A the set of qubits which H_1 acts upon, excluding q. Likewise, denote by B the set of qubits which H_2 acts on, excluding q. By the definition of the $\triangleright\triangleleft$ relation, we have $A \cap B = \Phi$. We can consider all operators in A as one qudit. Similarly, we can consider all qubits in B as another qudit. We can then apply lemma 3.4 and conclude that there exists a direct-sum decomposition of q that is preserved by both operators. The reason the decomposition of lemma (3.4) must be non-trivial is that otherwise the decomposition is to a sum of zero and two-dimensional spaces, since $dim(q) = 2$, which means that one of the operators acts trivially on q, contradicting our assumption. ■

As mentioned in the introduction, we notice as a first step, that if there are two butterflies with respect to q, $H_1(q) \triangleright\triangleleft H_2(q)$ and $H_1(q) \triangleright\triangleleft H_3(q)$, then due to the low dimensionality, the decompositions induced by both $\triangleright\triangleleft$ relations are the same:

Claim 4.4: **Unique butterfly induced decomposition of q.** Consider two butterflies $H_1 \triangleright\triangleleft H_2$, $H_1 \triangleright\triangleleft H_3$, both with respect to q, where all three operators act non-trivially on q. Then the decompositions induced on q from both butterflies must be the same.

Proof: As in Claim (4.3), q can be decomposed into a direct sum of two one dimensional subspaces, based on the first butterfly $H_1 \triangleright\triangleleft H_2$. Let Π_q^0, Π_q^1 be the projections on those subspaces of $\mathcal{H}_q$, so $\Pi_q^0 + \Pi_q^1 = I$. We can write

$$H_1 = \Pi_q^0 \otimes \Pi_A^0 + \Pi_q^1 \otimes \Pi_A^1 \tag{5}$$

where Π_A^i are some projections on A, by lemma (3.4). Similarly, a non-trivial decomposition exists for the second relation $H_1 \triangleright\triangleleft H_3$.

$$H_1 = \tilde{\Pi}_q^0 \otimes \tilde{\Pi}_A^0 + \tilde{\Pi}_q^1 \otimes \tilde{\Pi}_A^1 \tag{6}$$

One can easily check that if $\tilde{\Pi}_q^0 \neq \Pi_q^0$ then this implies that $\tilde{\Pi}_A^0 = \tilde{\Pi}_A^1$ and so H_1 is in fact not dependent on q at all, contrary to our assumption. ■

This yields an important transitivity conclusion: i.e., if $H_1(q)$ and $H_2(q)$ agree on some decomposition of q, and $H_1(q)$ and $H_3(q)$ agree on some decomposition of q, then $H_2(q)$, and $H_3(q)$ agree on the same decomposition. We can now talk about two operators on q which are connected by a path of such butterflies: two operators are said to be $\triangleright\triangleleft$ connected (read this "butterfly-connected") if there is a sequence of $\triangleright\triangleleft$ relations that connects them, i.e. $H_1 \triangleright\triangleleft H_{i,1} \ldots \triangleright\triangleleft H_{i,m} \triangleright\triangleleft H_2$. A basic tool in this paper is the following theorem:

Theorem 4.5: If all pairs of operators $H_1(q), H_2(q)$ acting on a qubit q are butterfly connected, then q is separable.

Proof: Pick one operator acting on q, and now use claim (4.4) along the path connecting it to any other operator on q, to show that by transitivity all butterflies along the path induce the same decomposition on q. ■

Corollary 4.6: **"left-right" Partition implies Separability:** If there is a partition of the operator set S into two disjoint non-empty sets $S_{q,left}$ and $S_{q,right}$ such that for each $H_i \in S_{q,left}$ and $H_j \in S_{q,right}$ we have $A_i \cap A_j \subseteq \{q\}$ (we call this a "left-right" partition), then q is separable.

Proof: For any pair of operators H_1 and H_2 one can construct a chain of $\triangleright\triangleleft$ relations $H_1 \triangleright\triangleleft H_{j_1} \triangleright\triangleleft \ldots \triangleright\triangleleft H_{j_m} \triangleright\triangleleft H_2$ that goes back and forth between $S_{q,left}$ and $S_{q,right}$ as both sets are nonempty. Then by theorem (4.5) we get that q is indeed separable. ■

This theorem has an important implication.

Definition 4.7: **Operator Paths.** An operator path is an ordered set of L distinct operators $H_1, \ldots, H_L$ such that any contiguous two intersect by two qubits, and two operators that have more than one operator between them do not intersect.

Based on corollary (4.6) we show:

Claim 4.8: **Graph Connectivity implies operator path connectivity:** If S is a set of operators such that no qubit in A_S is separable, and G_S is connected, then A_S is also operator-path-connected, i.e., any pair of qubits $q, v \in A_S$, are connected by an operator path which starts with an operator which acts on q and ends with an operator which acts on v.

Proof: **Sketch:** Start from $q \in A_S$ and greedily add operators which are connected to the operators acting on q by an operator path. If we failed to reach v, there must be a qubit along a path from q to v which is separable by corollary (4.6). We omit the details. ■

This implies that an interaction graph made of non-separable qubits is severely limited, since its operators cannot be all connected by butterfly paths. The operators on any qubit thus cannot "fan-out" too much, as this would induce pairwise $\triangleright\triangleleft$ paths and would make this qubit separable.

We make this intuition more tangible, and show two important conclusions from Theorem (4.5). First, we define an "operator crown" on q, which is a set of three operators acting on q organized as in Figure 3.

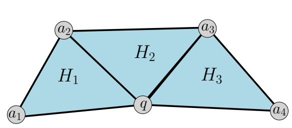

Figure 3. An operator crown on qubit q.

We show (using Theorem 4.5 and simple case by case analysis) that operator crowns act as "qubit traps";

Claim 4.9: **Operator Crown as an Operator Trap:** Let q be a nonseparable qubit, and let C be some operator crown on q. Then any operator on q acts on some crown qubit of C.

We omit the details of the proof.

Second, we show that any two operators on q must either intersect on one other qubit, or they are connected through another operator H_x, which intersects each of them with q and another qubit.

Claim 4.10: **Operators on q are connected by length-three operator paths:** Let q be some nonseparable qubit. Then any 2 operators on q are operator-path connected by an operator path on q of length ≤ 3.

Proof: The proof is based on a simple case by case analysis and Claims (4.8),(4.9), and Theorem (4.5). We omit the details. ■

These latter two properties impose severe restrictions on the geometry of the interaction graph.

4.2. A global 1D structure: The backbone

Having characterized the local geometric behavior of each individual nonseparable qubit in the residual graph, we are ready to make some claims w.r.t. the global structure of this graph. To this end, we define the "backbone" of the graph:

Definition 4.11: **Backbone:** For an instance S of $CLH(3,2)$ we define the backbone to be a maximal length operator path B in the connectivity graph G_S. If there are several such maximal length paths, we take one of them arbitrarily.

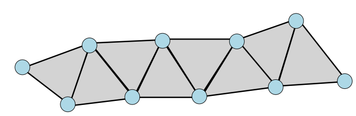

Figure 4. An operator backbone.

Intuitively, the backbone constitutes the longest possible stretch of "operator crowns" that are attached back to back, without revisiting qubits that have already been visited. Since we showed that an operator crown on qubit q essentially "traps" at least one other qubit of any operator acting on q, it follows that any operator that acts on a backbone qubit, must act on at least one more backbone qubit which is not very "far" in terms of backbone edges; in fact, roughly speaking, it must be a constant number of edges away. We call this **Short range connectivity in the backbone**.

This "qubit" entrapment property alone is not sufficient, however, for our purposes, as it does not handle operators that do not act on backbone qubits. We show that *all* operators must examine at least two backbone qubits, and moreover, these qubits cannot be too far away. Finally, we show that there are **no "shortcuts"** between far away qubits in the backbone, in the sense that no two operators that touch far away qubits in the backbone, intersect (not even through a qubit outside the backbone).

The result of all this is the following: Consider a coarse-graining of the backbone, in which say consecutive sets of 20 qubits are aggregated together and are considered as one particle of constant dimension; denote those by $\{Q_i\}$. By the above arguments, all interactions inside the backbone are two-local, namely, interact only Q_i and Q_{i+1}; and moreover, any qubit outside the backbone may interact only with a specific pair of consecutive large particles Q_i, Q_{i+1}. Denoting by V as the set of all qubits outside the backbone, we can thus denote a partition of V, $V = \sqcup_i V_i$ such that the following holds: for any $H_k \in S$ there exists an index i such that $A_k \subseteq \{Q_i, Q_{i+1}, V_i\}$.

This amounts to the following picture:

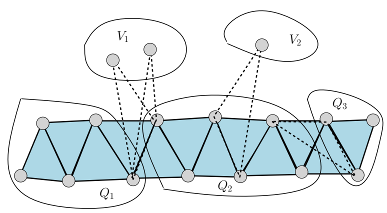

Figure 5. The interactions with the backbone: a backbone of sets Q_i - each comprised of a constant number of qubits, such that each operator acts on V_i and its associated pair of qudits Q_i, Q_{i+1}. We note that while the size of Q_i is constant, the size of V_i can be a function of n.

We examine this structured problem more closely. Consider the operators interacting Q_i with the qudit to its left, Q_{i-1}. Consider also the operators that interact Q_i with the qudit to its right, Q_{i+1}. We have a $\bowtie$ relation between any operator acting on Q_i from the left and any operator acting on Q_i from the right. We can then show, very similarly to [10], that there exists a decomposition of the Hilbert space of Q_i, such that when we restrict all operators on Q_i to a specific subspace in this decomposition, then Q_i can be written as a tensor product of two subparticles, the left subparticle $Q_{i,left}$ and the right subparticle $Q_{i,right}$, and the operators acting on Q_i from the left (right) interact only with the left (right) subparticle of Q_i. This paves the way for achieving two-locality: After partitioning each Q_i into those two separate subparticles $Q_{i,left}$ and $Q_{i,right}$, we can fuse the right side of one particle with the left side of the next: $Q_{i,right}$ with $Q_{i+1,left}$.

The resulting problem is two-local: all interactions are of the form in which one fused particle and one particle out of the backbone interact, or they are 1-local; hence, we get that each fused particle is a center of a star, and the stars are non-intersecting. This is already a problem in NP by the methods of [10].

5. Three-wise interactions for qutrits

We give here a rough sketch of the proof of theorem (1.2). Once again, we start by removing seperable qutrits, where separability means non-trivial decomposition of the qutrit space which is agreed upon by all operators. However, in the case of qutrits, operators may be butterfly connected with respect to q, and yet will not agree on a common decomposition of q. Hence, no equivalent of Theorem 4.5 holds, which destroys the basis for most of the geometric structure we managed to prove in the case of qubits.

To proceed, we weaken the definition of separability. To this end we define the notion of a **Critical Subspace** of an operator on a qudit. This is a one dimensional subspace, which exists in the decomposition of the induced algebra of the operator on the qubit. The notion of critical subspaces is what replaces the notion of unique decomposition in qubits, though it is weaker; we show that a $\bowtie$ relation between a pair of operators acting on q implies that each of the operators has a critical subspace; these critical subspaces are either orthogonal or identical, and moreover, the operators preserve each other's critical subspaces.

Using this notion, we can prove the following, when restricting the interactions to act on the plane: Consider all operators on a qutrit q; each operator has its own critical subspace in the Hilbert space of q. We prove that if the number of operators is large enough, any assignment of critical subspaces to the operators on q forces all of the operators to preserve at least one of the assigned critical subspaces, and so the qutrit becomes separable. This means that there cannot be more than a small number of operators acting on q, assuming q is not separable.

An easy implication of this property is the crucial fact that in an instance with no separable qutrits, all vertices in the interaction graph must be of degree at most 5.

Claim 5.1: Let S be an instance of $CLH(3,3)$ with no separable qudits. Then in the interaction graph of S, each qudit has degree at most 5.

Proof: **Sketch.** Suppose on the negative that q is a qudit of degree at least 6. Then it is acted upon by an operator path of length at least 5 (and possibly some other operators). Let $H_1, \dots H_5$, be that path, i.e. H_i, H_{i+1} intersect on q and one other qudit, for all $1 \le i \le 4$. Let us examine H_1, H_3, H_5. These operators share only q, and so they all agree on some non-trivial decomposition of q, which includes at least one 1-dim. subspace. Any other operator $H(q)$ aside from these, has a $\bowtie$ relation with at least one of H_1, H_3, H_5 since it is 3-local. Thus it must preserve the decomposition of these 3, and in particular it must preserve the 1-dimensional subspace of that decomposition. Thus, all operators on q agree on some non-trivial decomposition of q, contrary to q being nonseparable. ■

The main technical effort now is to show that planar-embedded Hamiltonians in which all the vertices are of degree at most 5, must exhibit an intriguing characteristic, which is in fact entirely geometrical. Consider a planar embedding of a graph, whose faces are colored black and white. Only 3-vertex faces that correspond to terms in the Hamiltonian are colored black. Then there must be a constant density of white "holes"; i.e., any point in the plane is within a constant distance (in terms of number of faces) from such a white hole - i.e., a region where no interaction acts.

Claim 5.2: Let S be an instance of $CLH(3,3)$ with no separable qudits, and let $G(S)$ be its interaction graph. Then there exists a constant η, such that for all such instance S, every face of $G(S)$ is at distance at most η from some "hole", i.e. a qudit triple on which there is no operator.

Proof: **Sketch.** The proof is purely geometrical, rather involved, and is based on Euler formula. We cannot provide the details due to lack of space. ∎

To deduce from this fact a way to coarse-grain the particles so that the interactions become two-local, we need to restrict the planar graph:

Definition 5.3: **Nearly-Euclidean (NE) Triangulation of a Polygon** A finite planar graph is said to be a Nearly Euclidean triangulation of a polygon if every face except the infinite face has three edges, the edges are straight lines, and moreover, the ratio between the shortest and longest edge is bounded from above by some overall constant, and the angle between any two incident edges is bounded from below by some overall constant angle.

The main point is that the existence of the regularly spaced holes, using claim (5.2) allows us, in the case in which the interaction graph is NE (and this is the only place where we use the NE property in the proof) to coarse-grain the set of particles, and by this derive a 2-local instance. The rough idea is to lay down on the plane a "net" that partitions the plane in such a way that in each region, there are only constantly many particles, while making sure that the junctions of the net fall precisely inside those white "holes". If we combine the particles in each region together, then each term in the Hamiltonian acts on at most 2 of the combined particles, and once again we can apply the 2-local methods of [10].

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